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SELF-PRESERVATION OF SLIGHTLY HEATED SMALL PERTURBATION JETS AND WAKES IN A PRESSURE GRADIENT

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NOMENCLATURE

C_1, C_2, C_3, C_4, C_5, constants defined in text;
$e(\eta), e_\theta(\zeta)$, self-preserving functions defined by equations (10) and (31);
$f(\eta), f_\theta(\zeta)$, self-preserving functions defined by equations (2) and (22);
$g(\eta), g_\theta(\zeta)$, self-preserving functions defined by equations (3) and (23);
$h(\eta), h_\theta(\zeta)$, self-preserving functions defined by equations (10) and (31);
$k(\eta), k_\theta(\zeta)$, self-preserving functions defined by equations (10) and (31);
l_0, l_θ, scaling lengths for the velocity and the temperature, respectively;
m, exponent for the x variation of U_1, equation (15);
n, exponent for the x variation of u_0, equation (16);
p, exponent for the x variation of l_0, equation (16); also kinematic fluctuating pressure, equation (9);
q^2, $u^2 + v^2 + w^2$;
T, mean temperature;
U, V, mean velocities in the x and y directions;
U_1, free stream velocity;
u, v, w, fluctuating velocities in the x, y and z directions;
u_0, velocity scale;
x, axial distance;
y, distance normal to axis of symmetry.

Greek symbols

α, thermal diffusivity of fluid;
α_1, β, γ, exponential indices defined in text;
$\varepsilon, \varepsilon_\theta$, mean dissipation of turbulent energy, equation (9), and of temperature, equation (31);
η, y/l_0;
ζ, y/l_θ;
θ_0, temperature scale;
ν, kinematic viscosity of fluid.

Others

prime, denotes derivative with respect to the argument of the function;
overbar, denotes time average.

GARTSHORE and Newman [1] have shown that the approximately self-preserving isothermal jet or wake in zero pressure gradient is only a particular example of a class of approximately self-preserving flows in pressure gradients. Necessary conditions for self-preservation in both two-dimensional and axisymmetric flows were obtained from the mean flow momentum equation and the turbulent energy equation. Townsend [2] showed that if small amounts of heat are present in a turbulent flow that is developing in self-preserving fashion, the temperature distribution may also be of self-preserving form. In particular, when the velocity increment in the case of a jet or the deficit in the case of a wake is proportional to the velocity U_1 of the streaming flow, the development is possible only if the temperature scale θ_0 is proportional to the velocity increment (or deficit). In this note, it is shown that the temperature length scale is proportional to the velocity length scale l_0 for the case of a slightly heated two-dimensional (or axisymmetric) jet and wake in which the velocity increment or deficit is small compared with U_1. Also, the streamwise variation of θ_0 is obtained and the condition for exponential variation of θ_0 will be made precise. With $U_1 \sim x^m$, the bounds on m for a two-dimensional flow are given [1] by $-1/3 \le m \le 0$. In particular, when $m = -1/3$, the length scale l_0 varies linearly, the velocity scale U_1 varies as $x^{-1/3}$ and θ_0 varies as $x^{-2/3}$. In a zero pressure gradient (e.g. Newman [3]), $l_0 \propto x^{1/2}$, $u_0 \propto x^{-1/2}$. It is shown here that, for the latter case, θ_0 is proportional to u_0.

The mean momentum equation for a two-dimensional flow can be approximated to

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial \overline{uv}}{\partial y} = U_1\frac{dU_1}{dx} + \nu\frac{\partial^2 U}{\partial y^2}. \qquad (1)$$

Using Townsend's [2] notation, the self-preservation forms for the velocity field are assumed to be given by

$$U = U_1 + u_0 f(\eta), \qquad (2)$$

$$\overline{uv} = u_0^2 g(\eta). \qquad (3)$$

The normal velocity V is obtained by integrating the continuity equation (assuming constant density), viz.

$$V = -l_0\frac{dU_1}{dx}\eta + u_0\frac{dl_0}{dx}\eta f - \frac{d}{dx}(u_0 l_0)\int_0^\eta f\,d\eta. \qquad (4)$$

For small perturbation jets and wakes, i.e. when $|u_0| \ll U_1$, (1) may be approximated, after substitution of (2), (3) and (4) and neglecting terms of order $O((u_0/U_1)^2)$, to

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$$C_1 f - C_2 \eta f' + g' = \frac{\nu}{u_0 l_0} f''. \quad (5)$$

When the Reynolds number $u_0 l_0/\nu$ is large, the right side of (5) may be neglected; consequently the coefficients C_1 and C_2 are given by

$$C_1 = \frac{l_0}{u_0^2} \frac{d}{dx}(u_0 U_1) = \text{const.} \quad (6)$$

$$C_2 = u_0^{-1} \frac{d}{dx}(l_0 U_1) = \text{const.}, \quad (7)$$

where the constancy is required for (5) to be consistent with the assumption of self-preservation. In particular, integration with respect to η of the reduced form of the mean momentum equation (5) leads to (see [1])

$$C_1 + C_2 = 0 \quad (8)$$

with the assumption that f approaches zero more quickly than η^{-1} as $|\eta|$ becomes large.

The turbulent kinetic energy equation may be written as

$$U \frac{\partial}{\partial x}\left(\frac{\overline{q^2}}{2}\right) + V \frac{\partial}{\partial y}\left(\frac{\overline{q^2}}{2}\right) + \overline{uv}\frac{\partial U}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\overline{q^2 v}}{2} + \overline{pv}\right) + \varepsilon = 0. \quad (9)$$

Assuming self-preserving forms of the type

$$\left.\begin{aligned} \frac{\overline{q^2}}{2} &= u_0^2 k(\eta), \\ \frac{\overline{q^2 v}}{2} + \overline{pv} &= u_0^3 h(\eta), \\ \varepsilon &= \frac{u_0^3}{l_0} e(\eta), \end{aligned}\right\} \quad (10)$$

(9) can be approximated to, neglecting terms of $0((u_0/U_1)^2)$,

$$2C_3 k - C_2 \eta k' + f'g + h' + e = 0, \quad (11)$$

where

$$C_3 = \frac{l_0 U_1}{u_0^2} \frac{du_0}{dx} = \text{const.}, \quad (12)$$

as required if the self-preserving distributions of (10) are to satisfy (9).

Equations (6), (7) and (12) can be combined to yield

$$\frac{d^2 U_1}{dx^2} + \left(\frac{1}{m} - 1\right) U_1^{-1} \left(\frac{dU_1}{dx}\right)^2 = 0, \quad (13)$$

where

$$m = \frac{C_1 - C_3}{C_2 - C_3}. \quad (14)$$

When $C_2 \neq C_3$, (13) admists a power-type solution

$$U_1 \sim (x - x_0)^m, \quad (15)$$

where x_0 can be identified as the hypothetical origin of the flow. Without loss of generality, x_0 will be put to zero in the subsequent development. Also, with the assumption that $C_2 \neq C_3$, it can be easily shown that l_0 and u_0 follow power-law distributions

$$u_0 \sim x^n, \quad l_0 \sim x^p \quad (16)$$

where

$$n = \frac{C_3}{C_2 - C_3}, \quad p = \frac{C_2 + C_3 - C_1}{C_2 - C_3}. \quad (17)$$

The physical constraint (8) requires that m, n, p are related through

$$n = -\tfrac{1}{2}(m + 1) \quad \text{and} \quad p = \tfrac{1}{2}(1 - 3m). \quad (18)$$

Gartshore and Newman [1] indicated that the exponent m lies in the range

$$-\tfrac{1}{3} \leq m \leq 0.$$

The lower limit (which actually corresponds to the exactly self-preserving wake or jet for which $u_0 \sim U_1$) is obtained from the requirement that $|u_0|/U_1$ should not increase in the x direction. The upper limit, which corresponds to a zero pressure gradient, was obtained by requiring the Reynolds number $u_0 l_0/\nu$ not to decrease with x. As pointed out by Gartshore and Newman [1], this latter limit is not likely to be restrictive for practical purposes.

Of interest is the case $C_2 = C_3$, where an exponential solution of (13) is possible

$$U_1 \sim e^{\alpha_1 x} \quad (19)$$

where α_1 is an arbitrary constant. For this case, l_0 and u_0 also follow an exponential scaling

$$u_0 \sim e^{\beta x}, \quad l_0 \sim e^{\gamma x} \quad (20)$$

where the constraint $C_1 = -C_2 = -C_3$ requires that

$$\beta = -\tfrac{1}{2}\alpha_1 \quad \text{and} \quad \gamma = -\tfrac{3}{2}\alpha_1. \quad (21)$$

This exponential behaviour has not received much attention from an experimental point of view.

When only a small amount of heat is added to the fluid, the dynamics of the flow should not be affected and the previous conditions on U_1, u_0 and l_0 for a self-preserving development of the flow remain, of course, unaltered. It is assumed that the mean temperature T relative to the ambient temperature (here assumed constant* with respect to x) of the external stream has a self-preserving form

$$T = \theta_0 f_\theta(\zeta). \quad (22)$$

Note that we do not *a priori* require the temperature length scale $l_\theta(x)$ to be proportional to the velocity length scale $l_0(x)$.

With the heat flux $\overline{v\theta}$ given by

$$\overline{v\theta} = u_0 \theta_0 g_\theta(\zeta), \quad (23)$$

the mean enthalpy equation

$$U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} + \frac{\partial \overline{v\theta}}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}, \quad (24)$$

leads to, neglecting terms of $0((u_0/U_1)^2)$,

$$\frac{U_1 l_\theta}{u_0 \theta_0}\frac{d\theta_0}{dx} f_\theta(\zeta) - \frac{1}{u_0}\frac{d}{dx}(l_\theta U_1)\zeta f'_\theta(\zeta) + g'_\theta(\zeta) = \frac{\alpha}{u_0 l_\theta} f''_\theta(\zeta),$$

$$C_4 f_\theta(\zeta) - C_5 \zeta f'_\theta(\zeta) + g'_\theta(\zeta) = 0 \quad (25)$$

when the Péclet number $u_0 l_0/\alpha$ is large.

For self-preservation, C_4 and C_5 must be constant. Integration of equation (25) across the flow leads to, assuming that $f_\theta(\zeta)$ approaches zero more rapidly than ζ^{-1} at large $|\zeta|$,

$$C_4 + C_5 = 0. \quad (26)$$

For the case where U_1, u_0 and l_0 admit power-type solutions ($C_2 \neq C_3$)

$$l_\theta \sim x^p, \quad \theta_0 \sim x^{C_4}, \quad (27)$$

where p is given by equation (17) and the physical constraint (26) requires that

*Townsend [2] points out that if the ambient fluid is at rest, the ambient temperature need not be constant and, for self-preservation of a simple jet, the ambient temperature gradient could vary as a power of x.

$$\theta_0 \sim (l_\theta U_1)^{-1},$$

or

$$\theta_0 \sim x^{1/2(m-1)}. \quad (28)$$

Thus the temperature length scale is proportional to the velocity length scale $l_0(x)$, and $\theta_0 \sim (u_0 U_1)$.

The lower and upper limits of m correspond to $\theta_0 \sim x^{-2/3}$ and $\theta_0 \sim x^{-1/2}$, respectively. This latter variation coincides with that which applies to a heated plane turbulent jet (e.g. Davies *et al.* [4]) for which $l_0(\sim l_\theta)$ varies approximately linearly. It should be noted that Townsend's conclusion [2] that the temperature scale must be proportional to the velocity scale ($\theta_0 \sim u_0$) is incorrect due to an erroneous statement of the conservation of momentum (for $|u_0| \ll U_1$).

For the case where $C_2 = C_3$, U_1, u_0 and l_0 admit exponential type solutions and it can be shown that

$$l_\theta \sim e^{-3/2(\alpha_1 x)} \sim l_0, \quad (29)$$

$$\theta_0 \sim e^{1/2(\alpha_1 x)} \sim u_0 U_1. \quad (30)$$

Again, the conclusions of the previous paragraph remain valid for this case.

It is worthwhile to enquire whether the equation for the intensity of temperature fluctuation analogous to equation (9) is satisfied by the self-preserving distributions of T, $\overline{v\theta}$ and

$$\left.\begin{aligned} \tfrac{1}{2}\overline{\theta^2} &= \theta_0^2 k_\theta(\eta), \\ \tfrac{1}{2}\overline{\theta^2 v} &= u_0\theta_0^2 h_\theta(\eta), \\ \varepsilon_\theta &= \frac{u_0\theta_0^2}{l_0} e_\theta(\eta), \end{aligned}\right\} \quad (31)$$

Note that, since $l_\theta \sim l_0$, no distinction is now made between ζ and η. The equation for $\overline{\theta^2}/2$

$$U\frac{\partial}{\partial x}(\tfrac{1}{2}\overline{\theta^2}) + V\frac{\partial}{\partial y}(\tfrac{1}{2}\overline{\theta^2}) + \overline{v\theta}\frac{\partial T}{\partial y} + \frac{\partial}{\partial y}(\tfrac{1}{2}\overline{\theta^2 v}) + \varepsilon_\theta = 0$$

reduces to, neglecting terms of $O((u_0/U_1)^2)$,

$$2C_4 k_\theta - C_2\eta k'_\theta + g_\theta f'_\theta + h'_\theta + e_\theta = 0. \quad (32)$$

Clearly, no new constraint emerges from (32) and the equation of the mean squared temperature fluctuation is satisfied by the assumed self-preserving forms in (31).

For an axisymmetric small-perturbation turbulent jet it is easy to show, using an approach analogous to that developed in [1] for the treatment of flow without heat transfer, that

$$\theta_0 \sim (U_1 l_0^2)^{-1} \sim u_0 U_1,$$

or

$$\theta_0 \sim x^{m-2/3} \quad (33)$$

since ([1])

$$l_0 \sim x^{1/3-m}$$

and $u_0 \sim x^{-2/3}$, irrespective of the value of m ($-\frac{2}{3} \le m \le -\frac{1}{3}$). There seems to be little if no experimental evidence available to support equations (28) or (33). The data obtained by Antonia and Bilger [5] in a heated round jet in a coflowing stream with no pressure gradient (the jet to external stream velocity ratio was 3) indicate that $\theta_0 \sim x^{-1}$ while (33) yields $\theta_0 \sim x^{-2/3}$. In this experiment, m is zero and therefore outside the range $-\frac{2}{3} \le m \le -\frac{1}{3}$ while the condition $|u_0| \ll U_1$ is not satisfied. At the last measurement station (the flow was still turbulent) $u_0 \simeq 0.15\,U_1$.

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INFLUENCE OF TRANSVERSE INTRAPHASE VELOCITY PROFILES AND PHASE FRACTION DISTRIBUTIONS ON THE CHARACTER OF TWO-PHASE FLOW EQUATIONS

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NOMENCLATURE

- a_{GL}, interfacial area per unit mixture volume;
- $A_{x\text{-}s}$, channel cross-sectional area;
- C_G, C_L, C_G^*, C_L^*, distribution coefficients (defined in text);
- $\bar{f}_k$, k-phase average of variable f $\left(\equiv \frac{1}{\Delta t_k}\int_{[\Delta t]_k} f_k\,dt\right)$;
- F_{Bk}, body force on k-phase per unit mixture volume, z-component;
- F_{Wk}, frictional drag force on k-phase per unit mixture volume due to channel wall;
- $\mathbf{g}_k$, body force per unit mass of k-phase;
- j, $\sqrt{-1}$;
- p_k, k-phase static pressure;
- r, radial coordinate;